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The curvature argument☆

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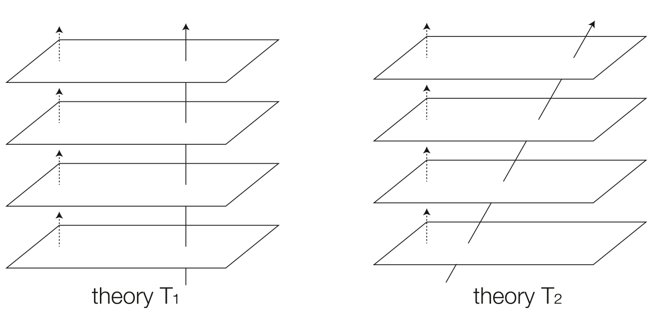
ABSTRACT

Dasgupta (2015) has recently put forward a novel argument, which he calls the ‘curvature argument’, that aims to show that Galilean spacetime is not an ideal setting for our classical theory of motion. This paper examines the curvature argument and argues that it is not sound. The discussion yields a remark about the conditions under which a ‘symmetry argument’ demonstrates that a particular spacetime is a non-ideal setting for our theory of motion.

1. Introduction

In its modern gloss, Leibniz’s ‘boost argument’ demonstrates that Newtonian spacetime is not an ideal spacetime setting for our classical theory of motion. Newtonian spacetime famously comes equipped with an ‘absolute standard of rest’. That is, it comes equipped with enough structure to distinguish between trajectories that are *at absolute rest* and trajectories that are not.

The boost argument proceeds as follows.[1] Suppose that we see some body that is free of forces and we want to know whether it is at rest or moving at some constant but non-zero velocity. We recognize, as illustrated in the figure below, that there are two theories about this free body that are both compatible with all of our observations. (Note that the dotted arrows in the figure represent the standard of rest that Newtonian spacetime is equipped with; ‘vertical’ trajectories are at rest. The solid arrows represent the trajectory of the body according to the theory.) The theory T_1 says that the body is at rest. The theory T_2 says that the body is moving at a constant non-zero velocity. One often says that this second theory has been obtained from the first by ‘boosting’ the velocity of the body.



The boost argument begins from the following three premises about these two theories.

P1. T_1 asserts that the body is at rest.
P2. T_2 asserts that the body is not at rest.
P3. We have no reason to prefer T_1 over T_2, or vice versa.

The claims P1 and P2 follow immediately from the way that the two theories have been described. For P3, we note first that the empirical data do not distinguish between these two theories. And moreover, we have no non-empirical reasons to prefer one of the theories over the other. They are, for example, equally parsimonious.

It is standard to use these three premises – in combination with some other supplementary premises – to conclude that there is something “unscientific and ‘bad’” about the Newtonian standard of rest (Friedman, 1983, p. 112). Different philosophers argue for different precise senses in which it is bad. Some claim that the Newtonian standard of rest is dispensable (Friedman, 1983), others that it is undetectable (Dasgupta, 2015, 2016), and yet others that it is surplus or redundant (Earman, 1989; North, 2009). For our purposes, however, the important thing is that there is *something* wrong with absolute rest and the premises P1, P2, and P3 play the crucial role in showing that. While there is disagreement about the exact kind of ‘badness’ that the concept of absolute rest exhibits, there is nonetheless a general consensus that it is bad in some sense, and therefore that some argument relying on P1, P2, and P3 suffices to show that Newtonian spacetime is not an ideal spacetime setting for our classical theory of motion.

Dasgupta (2015, p. 614–15) has recently put forward a novel argument, which he calls **the curvature argument**, that aims to show that *Galilean spacetime* – the spacetime that is obtained from Newtonian spacetime by excising the Newtonian standard of rest – is also not an ideal setting for our classical theory of motion. It too comes equipped with some structure that is unscientific and bad, in the same sense (whatever that may be) as the Newtonian standard of rest was.

The curvature argument proceeds in a similar manner to the boost argument. Suppose that we see some body that is moving free of

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[1] See for example Geroch (1972, p. 34), and also the presentations in Dasgupta (2015) and Maudlin (2012).

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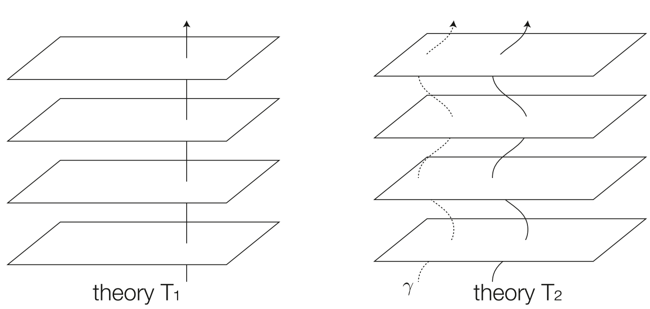
forces and we want to know whether this body is traversing a curved trajectory through spacetime or whether it is traversing a straight trajectory through spacetime. Dasgupta (2015, p. 615) describes the following two theories about this body that are compatible with all of our observations:

> T_1 picks out the straight trajectories as being 'privileged'; that is, as being those trajectories that are followed by bodies in the absence of force. But now pick a curved trajectory [the dotted line in the figure below], and take the set of all trajectories that are unaccelerated relative to it. According to T_2, they are the privileged trajectories followed by bodies in the absence of force, and bodies under the influence of force follow trajectories that are curved relative to those privileged ones (where the degree of curvature is again proportional to the force). Now, imagine that we see a body free of force. What does this indicate? According to T_1, it indicates that the body is making a straight trajectory; according to T_2, it indicates that it is making a certain kind of curved trajectory. So, if one does not know whether T_1 or T_2 is true, and all one sees is a body free of force, one will not know whether it is moving along a straight or curved trajectory! Assume now that we cannot know whether T_1 or T_2 is true. Then, we cannot detect whether a body is moving along a curved or straight trajectory.

The theory T_1 is our standard account – classical physics set in Galilean spacetime – of which trajectories through spacetime are traversed by free bodies. The theory T_2 presents an alternative account. It begins by picking some curved trajectory γ through spacetime, like the dotted line in the figure below, and then it asserts that free bodies traverse those trajectories that are non-accelerating relative to γ. So if we see a free body, we are not sure whether T_1 is true and that body is traversing a straight trajectory through spacetime or whether T_2 is true and that body is traversing a curved trajectory. This is meant to be precisely like the situation in Newtonian spacetime, where we could not say whether the body was at rest or moving at a constant non-zero velocity.



Like the boost argument, Dasgupta's curvature argument turns on the following three premises.

P1. T_1 asserts that the body is traversing a straight trajectory.
P2. T_2 asserts that the body is traversing a curved trajectory.
P3. We have no reason to prefer T_1 over T_2, or vice versa.

Insofar as one thinks that the boost argument shows that there is something wrong with the Newtonian standard of rest, one should also think that these three premises (if true) show that there is something wrong with the Galilean standard of 'straightness'. The curvature argument turns on exactly the same core premises as the boost argument did. If we agree that the boost argument shows that Newtonian spacetime is not an ideal setting for our classical theory of motion, then so long as its premises are true, the curvature argument will show that Galilean spacetime is not an ideal setting for our classical theory of motion either. It too comes equipped with some structure that is unscientific and bad in the same sense as the Newtonian standard of rest was.[2]

My aim in what follows is to argue that the curvature argument is of an importantly different character than the boost argument. In particular, it is not sound. Depending on how one precisely formulates the theory T_2, either P2 is false or P3 is false. In brief, if one formulates T_2 with enough structure to make P2 true, then P3 is false because T_2 posits a piece of 'surplus structure' and therefore strictly 'more structure' than T_1. This discussion will yield a broader remark about the conditions under which a symmetry argument, like the boost argument or the curvature argument, demonstrates that a particular spacetime is a non-ideal setting for a theory of motion.

2. The curvature argument

We begin by presenting the curvature argument in detail. In particular, we need to carefully formulate the two theories T_1 and T_2. Since it is the more straightforward of the two, we begin with T_1, the standard classical theory of motion set in Galilean spacetime.

Spacetime theories begin by specifying a smooth, connected, four-dimensional manifold M.[3] Each point $p \in M$ represents the location of an 'event' in spacetime. **Galilean spacetime** is the tuple $(\mathbb{R}^4, t_a, h^{ab}, \nabla)$. The smooth tensor fields t_a and h^{ab} and the derivative operator ∇ are defined as follows.

$$t_a = d_a x^1$$
$$h^{ab} = \left(\frac{\partial}{\partial x^2}\right)^a \left(\frac{\partial}{\partial x^2}\right)^b + \left(\frac{\partial}{\partial x^3}\right)^a \left(\frac{\partial}{\partial x^3}\right)^b + \left(\frac{\partial}{\partial x^4}\right)^a \left(\frac{\partial}{\partial x^4}\right)^b$$
∇ is the coordinate derivative operator on $\mathbb{R}^4$

Here $d_a x^i$ is the differential of the standard coordinate function $x^i : \mathbb{R}^4 \to \mathbb{R}$ and $\left(\frac{\partial}{\partial x^i}\right)^a$ is the standard ith coordinate vector field on $\mathbb{R}^4$. The coordinate derivative operator ∇ on $\mathbb{R}^4$ is defined to be the unique derivative operator that satisfies $\nabla_a\left(\frac{\partial}{\partial x^i}\right)^b = \mathbf{0}$ for each $i = 1, \ldots, 4$.[4] We note that ∇ is flat, in the sense that its curvature field $R^a{}_{bcd} = \mathbf{0}$ everywhere on $\mathbb{R}^4$.

These geometric structures on Galilean spacetime are interpreted as follows. The field t_a is a temporal metric. It assigns a temporal length to vectors, and defines a preferred partitioning of Galilean spacetime into simultaneity slices. The field h^{ab} is a spatial metric. Given a vector ξ^a, one can use h^{ab} to (indirectly) assign a spatial length to it. Finally, and most importantly for our purposes, the derivative operator ∇ endows $\mathbb{R}^4$ with a standard of constancy. It specifies which trajectories through Galilean spacetime are geodesics, or in other words, 'straight'. A curve $\gamma' : \mathbb{R} \to \mathbb{R}^4$ through Galilean spacetime with tangent field ξ^a is a geodesic just in case $\xi^n \nabla_n \xi^a = \mathbf{0}$, i.e. it is non-accelerating relative to ∇. The derivative operator ∇ is compatible with t_a and h^{ab}, in the sense that $\nabla_n t_a = \mathbf{0}$ and $\nabla_n h^{ab} = \mathbf{0}$. This simply means that, as one should expect of a classical spacetime, the temporal and spatial metrics are 'constant'. We note, however, there is in general more than one derivative operator that is compatible with t_a and h^{ab}. This is unlike the case in which a manifold has a *non-degenerate* metric on it, like a relativistic spacetime does. In that case there is a unique derivative operator compatible with the metric.

We have described the background spacetime setting, but we also need to state the equation of motion that T_1 employs. If a particle has

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[2] Dasgupta intends for this to be an argument against a view he calls 'Galilean substantivalism', "the view that Galilean space–time exists, and has this structure, independently of its material constituents" (Dasgupta, 2015, p. 613). If the curvature argument is sound, it would refute Galilean substantivalism by denying that second clause; spacetime would not have the kind of 'straightness' structure that Galilean spacetime posits.

[3] The reader is invited to consult (Malament, 2012) for details on the preliminaries that follow.

[4] For proof that the coordinate derivative operator is unique see Malament (2012, Prop. 1.7.11).

mass m, then T_1 says that it will traverse a smooth timelike curve whose tangent field ξ^a satisfies $t_a\xi^a = 1$ and

$$F^a = m\xi^n\nabla_n\xi^a \tag{1}$$

where F^a is the vector field representing the net force acting on the particle. The equation of motion (1) immediately implies that massive bodies that are free of forces travel along trajectories that are geodesics relative to ∇. So imagine that we are observing some body that is free of forces, and recall the premise P1 from the curvature argument.

P1. T_1 asserts that the body is traversing a straight trajectory.

The premise P1 is true. The standard theory T_1 does say that massive bodies experiencing no net force will traverse geodesics, or 'straight trajectories', through spacetime.

The theory T_2 requires a bit more work to make precise. In order to simplify matters, we will work with a representative case of T_2 by picking one particular smooth curve $\gamma : \mathbb{R} \to \mathbb{R}^4$. We consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^4$ defined by

$$\gamma(t) = (t, \sin(t), 0, 0)$$

in standard coordinates on $\mathbb{R}^4$. This choice of γ is perfectly representative – indeed, it matches the situation depicted in the figure above – and one imagines that it results in a canonical example of the kind of theory T_2 that Dasgupta has in mind. We will use γ to determine the new standard of non-acceleration for T_2. One can easily compute in standard coordinates that the tangent field λ^a and four-acceleration field $\lambda^n\nabla_n\lambda^a$ of γ are the following.

$$\lambda^a = \left(\frac{\partial}{\partial x^1}\right)^a + \cos(x^1)\left(\frac{\partial}{\partial x^2}\right)^a \qquad \lambda^n\nabla_n\lambda^a = -\sin(x^1)\left(\frac{\partial}{\partial x^2}\right)^a$$

Since the four-acceleration of γ is non-zero, it is indeed that case that γ is a 'curved' trajectory through Galilean spacetime, as Dagupta requires.

Now Dasgupta (2015, p. 615) uses γ to pick out a class of privileged curves that force-free bodies traverse. Recall that the way in which he proposes to do this is to "take the set of all trajectories that are unaccelerated relative to [γ]. According to T_2, they are the privileged trajectories followed by bodies in the absence of force, and bodies under the influence of force follow trajectories that are curved relative to those privileged ones (where the degree of curvature is again proportional to the force)". A particularly natural way to make this idea precise is as follows.

T_2 will be formulated using Galilean spacetime $(\mathbb{R}^4, t_a, h^{ab}, \nabla)$ as its background spacetime structure. But whereas the equation of motion (1) of T_1 said that the force on the body was proportional to its acceleration, the equation of motion of T_2 will say that the force on the body is proportional to its acceleration *relative to* γ, whose acceleration is itself non-zero. The following equation of motion makes this thought precise. If a particle has mass m, then T_2 says that it will traverse a smooth timelike curve whose tangent field ξ^a satisfies $t_a\xi^a = 1$ and

$$F^a = m\big(\xi^n\nabla_n\xi^a + \sin(x^1)\big(\frac{\partial}{\partial x^2}\big)^a\big) \tag{2}$$

where F^a is the vector field representing the net force acting on the particle.

This new equation of motion (2) is saying that the forces on a body dictate how 'accelerated' it is relative to γ. So as one can easily verify, if no forces are acting on the body, it will traverse a trajectory that has precisely the same acceleration as γ has. A net force on the body will make it traverse a trajectory that deviates from the state of acceleration that γ exhibits. This means that according to T_2, massive bodies that are free of forces travel along curved trajectories. To see this, suppose that a smooth curve γ' with tangent field ξ^a is the trajectory of a massive force-free body. Since the body is free of forces, $F^a = \mathbf{0}$ and Eq. (2) immediately implies that $\xi^n\nabla_n\xi^a = -\sin(x^1)\big(\frac{\partial}{\partial x^2}\big)^a$. So a curve γ' is a privileged curve according to T_2 if it has the same acceleration as γ. Or in other words, this is saying that the privileged curves through Galilean spacetime are those that are 'curved' just like γ is, according to the standard of constancy that ∇ determines.

We now have enough details on the formulation of T_2 to consider premise P2. Suppose that we observe some body that is not being acted upon by any forces.

P2. T_2 asserts that the body is traversing a curved trajectory.

There is a sense in which P2 is true. T_2 says that massive bodies experiencing no net force will traverse timelike curves whose 4-acceleration is the same as that of γ, which is itself non-zero. Such curves will therefore not be geodesics of ∇. Rather, they will be 'curved' relative to ∇, just like γ. On this understanding of T_2, therefore, there is a strong sense in which P2 is true.

On the face of it, P3 also strikes us as true.

P3. We have no reason to prefer T_1 over T_2, or vice versa.

Dasgupta (2015, p. 615) mentions the following considerations in favor of P3:

> They are empirically equivalent, so data are never going to refute one but not the other. Any reason to believe one over the other must therefore be based on some criteria other than empirical adequacy, such as simplicity. And, though T_1 is easier than T_2 to write down, it is not clear whether that is the kind of simplicity that yields an epistemic reason to believe T_1. Clearly, the issue depends on epistemological issues regarding what kinds of criteria yield reasons for belief.

Both of the theories will account for the phenomena equally well,[5] and on the face of it both of them seem to appeal to the same basic structures on spacetime: the spatial metric h^{ab}, the temporal metric t_a, and the derivative operator ∇. So unless we can isolate some good reason to prefer T_1 over T_2, or vice versa, P3 stands.

If it is the case that all of the premises P1, P2, and P3 are true, then the curvature argument will have the same standing as the boost argument. One can infer from these three premises that there is something 'bad' about the standard of 'straightness' – i.e. the derivative operator ∇ – that is associated with Galilean spacetime. It must be 'dispensable', 'redundant', 'surplus', or 'undetectable' structure, in the same sense as the Newtonian standard of absolute rest was. This is a surprising conclusion that dissents from the standard view that Galilean spacetime is an appropriate spacetime setting for classical physics.[6]

3. The curvature argument is unsound

The curvature argument, however, does not go through. Given the way we have formulated T_2, there is a strong sense in which P3 is false. T_2 comes equipped with more structure than T_1, in the form of an extra derivative operator in addition to ∇. It is therefore disingenuous to say – as we did when describing T_2 in the preceding section – that T_2 has Galilean spacetime as its background spacetime structure. Rather, T_2 appeals to strictly more structure than Galilean spacetime comes equipped with. A simple application of a structural parsimony principle therefore licenses us to prefer T_1 over T_2, rendering P3 false. It will take a moment to make this idea precise. Our first task is to precisely define this new derivative operator. We then demonstrate the sense in which T_2 appeals to it.

[5] That T_2 accounts for the phenomena as well as T_1 is implied by the results that follow. In particular, Proposition 3 implies that the equation of motion (2) is just a restatement of (1), but using a different derivative operator than ∇. This means that (2) is in fact just another way to state the standard Newtonian equation of motion.

[6] There are other notable dissenters, though they dissent for different reasons than Dasgupta. See, for example, the work of Saunders (2013) and Knox (2014) on how best to understand the geometry posited by Newton's theory of motion. See also Dewar (2018), Wallace (2017, 2020), and Weatherall (2016b). This debate takes place in the context of Newtonian gravitational theory, however, so the arguments put forward are of a different character than the curvature argument.

3.1. The derivative operator $f_*(\nabla)$

It will take a moment to describe the kind of additional derivative operator that T_2 appeals to. Suppose that $g : M \to N$ is a diffeomorphism between smooth manifolds M and N and that ∇ is a derivative operator on M. The diffeomorphism g allows us define a derivative operator $g_*(\nabla)$ on N by 'pushing forward' ∇. We define this new derivative operator by its action on an arbitrary smooth tensor field:

$$g_*(\nabla_n)\alpha^{a_1\ldots a_n}{}_{b_1\ldots b_m} = g_*(\nabla_n g^*(\alpha^{a_1\ldots a_n}{}_{b_1\ldots b_m}))$$

where $\alpha^{a_1\ldots a_n}{}_{b_1\ldots b_m}$ is an arbitrary smooth tensor field on N. The operator $g_*(\nabla)$ acts on $\alpha^{a_1\ldots a_n}{}_{b_1\ldots b_m}$ in an intuitive manner; $\alpha^{a_1\ldots a_n}{}_{b_1\ldots b_m}$ is first 'pulled back' onto M using g, then ∇ is applied to this tensor field on M, and the resulting tensor field is then 'pushed forward' onto N using g.

The following simple fact characterizes $g_*(\nabla)$. The proof of this lemma has been placed in the Appendix, along with proofs of all of the results that follow.

Lemma 1. *$g_*(\nabla)$ is a derivative operator on N. Moreover, a smooth curve γ' is a geodesic of ∇ if and only if $g\circ\gamma'$ is a geodesic of $g_*(\nabla)$.*

Since a derivative operator is fully characterized by its class of geodesics (Malament, 2012, Prop. 1.7.8), Lemma 1 tells us precisely what $g_*(\nabla)$ is like. It is the derivative operator whose geodesics are simply those of ∇ after they have been pushed forward onto N by the diffeomorphism g.

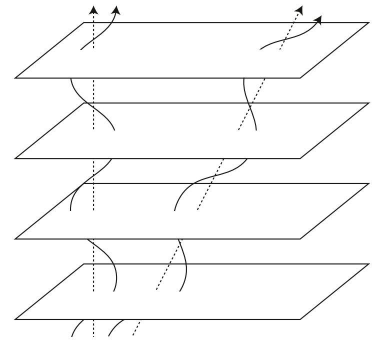
There is, of course, no requirement that M and N be distinct manifolds. If we have a diffeomorphism M to itself and a derivative operator on M, we can pushforward that derivative operator in the way just described in order to define another derivative operator on M. This is precisely what allows us to describe the extra derivative operator that T_2 is appealing to. It is the derivative operator $f_*(\nabla)$ on $\mathbb{R}^4$, where the diffeomorphism $f : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$f(x^1, x^2, x^3, x^4) = (x^1, x^2 + \sin(x^1), x^3, x^4)$$

in standard coordinates on $\mathbb{R}^4$. One can easily verify that f maps the 'x^1-axis' in Galilean spacetime to our privileged curve γ. Lemma 1 tells us exactly what standard of constancy the derivative operator $f_*(\nabla)$ is laying down on $\mathbb{R}^4$. Its geodesics are those trajectories that result from using f to uniformly 'bend' all of the straight curves (according to ∇) through Galilean spacetime. The resulting curves are then straight according to $f_*(\nabla)$. Since the curve $\gamma' : t \mapsto (t, 0, 0, 0)$ is a geodesic of ∇, Lemma 1 implies that $f\circ\gamma' = \gamma$, while not a geodesic of ∇, is a geodesic of $f_*(\nabla)$. The derivative operators ∇ and $f_*(\nabla)$ have different geodesics and are therefore distinct derivative operators on $\mathbb{R}^4$. But importantly, this new derivative operator still considers the spatial and temporal metrics on Galilean spacetime to be constant, as the following lemma demonstrates.

Lemma 2. *$f_*(\nabla)$ is compatible with both t_a and h^{ab}, in the sense that $f_*(\nabla_n)t_a = \mathbf{0}$ and $f_*(\nabla_n)h^{ab} = \mathbf{0}$.*

The following figure allows us to picture $f_*(\nabla)$. The dotted lines are geodesics relative to ∇. Their images under f, which are represented the solid lines, are geodesics of $f_*(\nabla)$.



Lemma 1 guarantees that all of the geodesics of $f_*(\nabla)$ result from 'bending' a geodesic of ∇ in this manner. In what follows we will show that these 'bent' curves – in other words, images of geodesics of ∇ under f – are the privileged trajectories that force-free bodies travel according to T_2.

3.2. T_2 appeals to $f_*(\nabla)$

There is a strong sense in which T_2 appeals to the derivative operator $f_*(\nabla)$. We will isolate three senses in which this is the case. First, the structures that T_2 explicitly appeals to naturally give rise to the further structure $f_*(\nabla)$, in much the same way as a manifold with metric naturally gives rise to its Levi-Civita derivative operator. Second, the structures that T_2 explicitly appeals to suffice to *implicitly define* $f_*(\nabla)$. And third, $f_*(\nabla)$ is appealed to by the T_2's equation of motion in exactly the same way as ∇ is appealed to by T_1's equation of motion. It will be useful to introduce a piece of notation. We define the class C of **privileged curves of** T_2 to consist of those timelike curves $\gamma' : \mathbb{R} \to \mathbb{R}^4$ whose tangent fields ξ^a satisfy $t_a\xi^a = 1$ and $\xi^n\nabla_n\xi^a = -\sin(x^1)\left(\frac{\partial}{\partial x^2}\right)^a$. The curves in C are those that have the same 4-acceleration as γ and are therefore possible trajectories of massive force-free bodies according to T_2.

We begin with a pair of lemmas. Recall that we say that a curve is **spacelike** if its tangent vectors have zero temporal length at every point. We can think of spacelike curves in Galilean spacetime as those that 'lie entirely in' a particular simultaneity slice. If $\gamma' : I \to M$ is a smooth curve with tangent field ξ^a, then we say that a tensor field $\alpha^{a_1,\ldots,a_m}_{b_1,\ldots,b_r}$ is **constant along** γ' (with respect to ∇) if $\xi^n\nabla_n\alpha^{a_1,\ldots,a_m}_{b_1,\ldots,b_r} = \mathbf{0}$.

Lemma 3. *If $\gamma' : \mathbb{R} \to \mathbb{R}^4$ is a smooth curve with tangent field ξ^a that satisfies $t_a\xi^a = 1$, then there is a unique vector field that agrees with ξ^a on the image of γ' and is constant along all spacelike curves.*

The idea behind Lemma 3 is clear. Galilean spacetime comes equipped with the structure necessary to naturally extend the tangent field of γ' to a vector field on all of $\mathbb{R}^4$. This extension proceeds in the following manner. Since γ' is guaranteed to intersect each simultaneity slice exactly once, each point in the simultaneity slice will be assigned the 'same' vector (according to the standard of constancy given by ∇) as the tangent field of γ' takes at the point where it intersects that slice. When no confusion will result, we will use the notation ξ^a to refer to the vector field whose existence is guaranteed by Lemma 3, and it will be understood that the field ξ^a is defined on all of $\mathbb{R}^4$ and not just on the image of γ'.

We need one more lemma before we can isolate the first sense in which T_2 appeals to $f_*(\nabla)$.

Lemma 4. *If γ' is a smooth curve in C with tangent field ξ^n, then the smooth tensor field $h^{ab} + \xi^a\xi^b$ is a metric on $\mathbb{R}^4$.*

We take a moment to unravel this result. Recall that while the spatial metric h^{ab} can be used to indirectly assign a spatial length to spacelike vectors (Malament, 2012, p. 253), it is *not* technically a metric on $\mathbb{R}^4$. It fails to satisfy the non-degeneracy condition on a metric, since there are non-zero covectors (like $d_a x^1$) that it assigns zero length to. This is important for the following reason. A non-degenerate metric on a manifold naturally determines a derivative operator on that manifold, in the sense that there is a unique derivative operator that is compatible with that metric. Since h^{ab} is not a non-degenerate metric, this result does not go through. As we mentioned earlier, there is more than one derivative operator that is compatible with both h^{ab} and t_a. Lemma 4 shows us, however, that each of the privileged curves in C gives rise to a non-degenerate metric on $\mathbb{R}^4$. Since each of these metrics – which will in general be distinct for different curves in C – will in turn give rise to a derivative operator on $\mathbb{R}^4$, this captures a sense in which each curve in C gives rise to a derivative operator on $\mathbb{R}^4$.

With these lemmas in hand, the first sense in which T_2 appeals to $f_*(\nabla)$ is captured by the following proposition. We show that each curve in C gives rise to the *same* derivative operator (via the procedure just described), and moreover, that derivative operator is $f_*(\nabla)$. The proof of this proposition, along with all the others, is contained in the Appendix.[7]

Proposition 1. *Let $\gamma' : \mathbb{R} \to \mathbb{R}^4$ be a smooth curve in C with tangent field ξ^n. Then $f_*(\nabla)$ is the unique derivative operator compatible with the metric $h^{ab} + \xi^a \xi^b$.*

This result is telling us that the structures that T_2 posits, in particular, those of Galilean spacetime plus the class C of privileged curves, naturally determine, give rise to, or 'define' the derivative operator $f_*(\nabla)$. This is in much the same way as any manifold with metric (M, g_{ab}) naturally comes equipped with its 'Levi-Civita' derivative operator. In that case there is simply a unique derivative operator that is compatible with g_{ab}. In the present case, there is a unique derivative that is compatible with all of the metrics that curves in C give rise to, and that derivative operator is $f_*(\nabla)$. More precisely, Proposition 1 guarantees that an arbitrary curve γ' in C gives rise to $f_*(\nabla)$ in three steps. First, by Lemma 3, the structure of Galilean spacetime allows us to naturally extends the tangent field ξ^a of γ' to all of $\mathbb{R}^4$. Second, by Lemma 4, this vector field ξ^a suffices to determine a metric on $\mathbb{R}^4$. And third, that metric in turn determines $f_*(\nabla)$, in the sense that $f_*(\nabla)$ is the unique derivative operator compatible with the metric. At the end of the day, therefore, Proposition 1 is capturing a robust sense in which the structures that T_2 posits include the derivative operator $f_*(\nabla)$.

This same idea can be captured in a slightly different manner. We will now show that the structures of Galilean spacetime plus the class C of privleged curves of T_2 suffice to *implicitly define* $f_*(\nabla)$, in the sense that any map that preserves the former structures also preserves the latter.[8] Recall that an **automorphism of Galilean spacetime** is a diffeomorphism $g : \mathbb{R}^4 \to \mathbb{R}^4$ that satisfies the following three conditions:

(i) $g^*(t_a) = t_a$,
(ii) $g^*(h^{ab}) = h^{ab}$,
(iii) $g_*(\nabla) = \nabla$, i.e. the two derivative operators agree in their action on all smooth tensor fields on $\mathbb{R}^4$.

The first two conditions require that g preserves the temporal metric t_a and the spatial metric h^{ab}, and the third requires that g preserves the derivative operator ∇. One can show that the third condition holds if and only if g 'preserves geodesics' in the following sense: a smooth curve $\gamma' : I \to \mathbb{R}^4$ is a geodesic with respect to ∇ if and only if $g \circ \gamma' : I \to \mathbb{R}^4$ is a geodesic with respect to ∇.[9]

We have the following result, which follows as a corollary to Proposition 1.[10]

Proposition 2. *Let $g : \mathbb{R}^4 \to \mathbb{R}^4$ be an automorphism of Galilean spacetime such that a smooth curve γ' is in C if and only if $g \circ \gamma'$ is in C. Then $g_*(f_*(\nabla)) = f_*(\nabla)$.*

After determining the symmetries of a particular mathematical object X, like Galilean spacetime plus the class C of privileged curves of T_2, it is standard practice in physics and mathematics to look for the structures on X that are 'invariant under' or 'preserved by' all of these symmetries. Those structures that are found to be invariant under the symmetries of X are often deemed to be 'determined by' or 'constructed from' or 'come for free given' the basic structures of X. Proposition 2 is telling us that $f_*(\nabla)$ is among those structures that are invariant under the symmetries of Galilean spacetime plus the class C. This means that $f_*(\nabla)$ is implicitly defined by the structures of Galilean spacetime – in particular, t_a, h^{ab}, and ∇ – plus the privileged class C of curves. And that captures our second sense in which T_2 appeals to $f_*(\nabla)$.

We now turn to the third and final way in which T_2 appeals to $f_*(\nabla)$. There is, in fact, a strong sense in which $f_*(\nabla)$ is doing all of the work for the theory T_2, while ∇ is doing none. The old derivative operator ∇ is essentially a piece of superfluous structure for T_2. The following result implies that the equation of motion that T_2 employs can be written in its most natural manner using $f_*(\nabla)$ rather than ∇.

Proposition 3. *Let $\gamma' : \mathbb{R} \to \mathbb{R}^4$ be a smooth curve with tangent field ξ^a that satisfies $t_a \xi^a = 1$. Then $\xi^n f_*(\nabla_n) \xi^a = \xi^n \nabla_n \xi^a + \sin(x^1) \left(\frac{\partial}{\partial x^2}\right)^a$.*

Proposition 3 is telling us that a curve's 4-acceleration according to $f_*(\nabla)$ is the same as its 4-acceleration relative to γ according to ∇. It implies that the class of privileged trajectories C that T_2 singles out is just the class of timelike geodesics (whose tangent fields satisfy $t_a \xi^a = 1$) of $f_*(\nabla)$. Proposition 3 also captures a strong sense in which T_2 appeals to $f_*(\nabla)$. Recall that if a body has mass m, then T_2 says that it will traverse a smooth timelike curve whose tangent field ξ^a satisfies $t_a \xi^a = 1$ and the equation of motion (2). Proposition 3 implies that Eq. (2) will hold of this curve if and only if

$$F^a = m \xi^n f_*(\nabla_n) \xi^a \tag{3}$$

holds of it, where once again F^a is the vector field representing the net force acting on the body. This restatement of T_2's equation of motion is just the standard equation of motion (1) that T_1 employed, but with the different derivative operator $f_*(\nabla)$ instead of ∇. So T_2 is appealing to the derivative operator $f_*(\nabla)$ in exactly the same manner as T_1 appealed to ∇: the most economical statement of the theory's equation of motion explicitly appeals to it. The derivative operator $f_*(\nabla)$ plays precisely the same role for the equation of motion of T_2 as ∇ played for the equation of motion of T_1.

The derivative operator $f_*(\nabla)$ captures the new standard of constancy that we defined by singling out the new class of privileged curves in our new equation of motion. The theory T_2 is therefore best understood as being formulated with the background spacetime structure $(\mathbb{R}^4, h^{ab}, t_a, \nabla, f_*(\nabla))$, which is clearly a strictly richer structure than Galilean spacetime. There is a strong sense in which objects that admit *more* structure-preserving maps between them have *less* structure. If there are more isomorphisms between objects of a certain type, then those objects intuitively must have less structure that these isomorphisms are required to preserve. Conversely, if there are has fewer isomorphisms between objects of some kind, then it must be

[7] See Malament (2012, Prop. 4.3.4) for a closely related result.

[8] See Barrett (2018) and Winnie (1986) for discussion of implicit definability.

[9] Barrett (2015b) and Weatherall (2016a) also define automorphisms of a classical spacetime in this way. See Barrett (2015b, Lemma 4) for proof that condition (iii) is the same as g preserving geodesics.

[10] This result is conceptually similar to the famous result of Malament (1977), who showed that the standard observer-relative simultaneity relation on Minkowski space–time is the only non-trivial equivalence relation that is invariant under maps that preserve the structures of Minkowski spacetime. This is standardly taken to capture a sense in which the theory comes equipped with a privileged notion of observer-relative simultaneity.

that those objects have more structure that the isomorphisms are being required to preserve. The amount of structure that an object has is, in some sense, inversely proportional to the number of symmetries that the object admits. Earman (1989, p. 36) puts this basic idea as follows: "As the space–time structure becomes richer, the symmetries become narrower". And North (2009, p. 87) writes that "stronger structure [. . .] admits a smaller group of symmetries". A collection of precise criteria for comparing amounts of structure have recently been proposed, all of which appeal to this basic idea that more structure-preserving maps should indicate less structure.[11] One can verify that these precise criteria for comparing 'amounts of structure' unsurprisingly judge $(\mathbb{R}^4, h^{ab}, t^{ab}, \nabla, f_*(\nabla))$ to have more structure than Galilean spacetime. Consider the 'temporal shift' $g : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$g : (x_1, x_2, x_3, x_4) \longmapsto (x_1 - \pi, x_2, x_3, x_4)$$

One can easily verify that g is an automorphism of Galilean spacetime. It does not, however, preserve the class of curves C. One computes that $g \circ \gamma(t) = (t, \cos(t), 0, 0)$, which clearly has a different 4-acceleration from γ. This means that g is not an automorphism of $(\mathbb{R}^4, h^{ab}, t^{ab}, \nabla, f_*(\nabla))$. Galilean spacetime therefore admits a broader class of symmetries than the spacetime structure presupposed by T_2, and therefore it has less structure than what is being posited by T_2.

This verdict is not surprising. Recall that in order to define the theory T_2 we first had to arbitrarily choose a smooth curve γ that would be the basis for our new standard of acceleration. When we made this decision and then used γ to single out the new class C of privileged curves, we were equipping T_2 with a new piece of structure. We can now return to premise P3 and see why it is false. Recall what P3 told us about T_1 and T_2:

P3. We have no reason to prefer T_1 over T_2, or vice versa.

We have seen here that we do have good reason to prefer the standard theory T_1 over T_2. T_2 posits strictly more structure than T_1 does. And moreover, T_2 comes equipped with a piece of structure that is superfluous. As long as one has $f_*(\nabla)$, one does not need ∇ to run the theory. As Proposition 3 demonstrates, the equation of motion for T_2 can be formulated without appeal to ∇. And this strikes one as exactly the kind of simplicity that gives us good reason to prefer T_1 over T_2. The two theories T_1 and T_2 are not on equal footing. If we want structurally simpler theories and theories that do not come equipped with superfluous structure, then we should prefer T_1 over T_2, so P3 is false.

4. The curvature argument revisited

This discussion of the relative merits of T_1 over T_2, however, suggests an alternative way to formulate T_2 that might allow the curvature argument to go through. In the formulation of T_2 presented in Section 2 above, the 'extra' derivative operator $f_*(\nabla)$ was implicitly appealed to. Proposition 3, however, showed us that the original derivative operator ∇ was actually not required in order to state the equation of motion (2) of T_2. This suggests that we should formulate T_2 without appeal to ∇, and that might avoid the kind of argument raised earlier — namely, that our initial formulation of T_2 posited more structure than T_1, rendering P3 false.

We take a moment to formulate precisely this alternative version of T_2. It uses the background spacetime structure $(\mathbb{R}^4, t_a, h^{ab}, f_*(\nabla))$. Since $f_*(\nabla)$ is compatible with t_a and h^{ab} (Lemma 2), this is a classical spacetime in the sense of Malament (2012). This spacetime no longer has 'more structure' than T_1; indeed one can verify that this spacetime is isomorphic to Galilean spacetime (and, indeed, that f is an isomorphism). There is therefore a strong sense in which the two spacetime posit precisely the same structure. So the argument against P3 that we put forward above will no longer go through. We can use the discussion surrounding Proposition 3 to formulate the equation of motion of T_2 in the following manner. If a particle has mass m, then this formulation of T_2 says that it will traverse a timelike curve whose tangent field ξ^a satisfies with $t_a \xi^a = 1$ and

$$F^a = m\xi^n f_*(\nabla_n)\xi^a \tag{3}$$

where F^a is again the vector field representing the net force acting on the particle. Proposition 3 implies that precisely the same trajectories satisfy this equation of motion as satisfied the equation of motion for our original formulation of T_2.

We now have a new formulation of T_2 that does not posit more structure than T_1. It may very well be that P3 is true in this case. But unfortunately, if we formulate T_2 in this way, then P2 is false. Suppose that we observe some body that is not being acted upon by any forces. Recall what P2 tells us about this body.

P2. T_2 asserts that the body is traversing a curved trajectory.

Under our current understanding of T_2, this is false. The new formulation of T_2 does not say that the body is traversing a curved trajectory. The only structure that T_2 can appeal to in order to classify a curve as 'straight' or 'curved' is $f_*(\nabla)$. And since $F^a = \mathbf{0}$, the equation of motion (3) of T_2 implies that the curve that this body is traversing will be a geodesic of $f_*(\nabla)$, so it is 'straight' according to $f_*(\nabla)$. This formulation has no other derivative operator to appeal to in order to classify the curve as 'curved'. If this is how we formulate T_2, P2 is false.

5. Symmetry arguments

We have therefore shown that the curvature argument does not demonstrate that there is something wrong with the standard of 'straightness' that Galilean spacetime employs. We presented two precise ways to formulate the theory T_2 that the curvature argument relies upon. In each case the curvature argument fails. When T_2 is formulated in the first way, we have good reason to prefer the standard theory T_1 since T_2 ends up coming equipped with surplus structure, rendering premise P3 false. When T_2 is formulated in the second way, it does not actually assert that the trajectories traversed by massive free bodies are curved, since the theory no longer comes equipped with ∇ — the piece of structure that allowed us to classify these trajectories as curved. And so in this case premise P2 is false. This response to the curvature argument is best put as follows: If one formulates T_2 with enough structure to make P2 true, then P3 will be false. In either case, the curvature argument is not sound.

We conclude by making a brief remark about the conditions under which a 'symmetry argument', like the boost argument or curvature argument, will succeed in demonstrating that a particular spacetime is a non-ideal setting for our theory of motion. In order to do this we need to isolate the conceptual difference between the boost argument and the curvature argument.

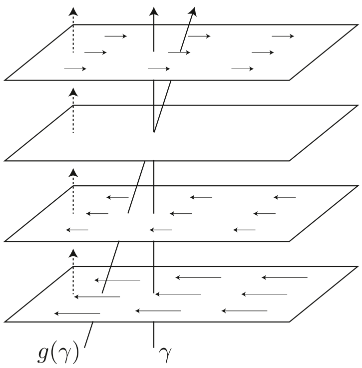
We begin by recalling the boost argument. Newtonian spacetime is the tuple $(\mathbb{R}^4, t_a, h^{ab}, \nabla, \lambda^a)$, where the temporal metric, spatial metric, and derivative operator are defined in precisely the same way as in Galilean spacetime, and the smooth vector field

$$\lambda^a = (\frac{\partial}{\partial x^1})^a$$

is the 'rigging' that determines the Newtonian standard of rest. The smooth timelike curves that have tangent field λ^a, i.e. those that are 'vertical' in the figure below, are the ones traversed by bodies that are at rest. The boost argument asks us to imagine a situation in which we observe a body that is free of forces. Given that we take Newtonian spacetime to best represent the underlying structure of spacetime, there are two theories about this body that are compatible with the empirical data (recall the figure from the introduction): according to T_1 the body

[11] See the literature on 'amounts of structure', e.g. Barrett (2015a, 2015b, 2021), Bradley (2020), North (2009), Swanson and Halvorson (2012), and Weatherall (2016c) for discussion.

is at rest, and according to T_2 the body is moving at some constant non-zero velocity. These two theories are related by a 'boost symmetry', which one can picture as follows.



The Newtonian standard of rest λ^a is represented in the figure by the dotted arrows on the left. The symmetry is represented by the small horizontal arrows that appear on each of the simultaneity slices. A boost symmetry is defined by 'flowing' each spacetime point along the vector field that the small horizontal arrows represent.[12] Formally, the boost symmetry is a diffeomorphism $g : \mathbb{R}^4 \to \mathbb{R}^4$ which, as Geroch (1978) sometimes put it, 'bevels the deck'. The thought is that one can picture the spacetime as a deck of cards, where each card represents a simultaneity slice, and a boost symmetry evenly 'bevels' this deck. We can think of g as uniformly boosting the velocity of every body in spacetime. The sense in which the boost symmetry relates our two theories about the motion of the body is the following. Applying the symmetry g to the trajectory of the body according to theory T_1 is how one obtains T_2. T_1 says that the body traverses the curve γ, which is at rest, while T_2 says that the body traverses $g(\gamma)$, which is not at rest.

It is important to note the following fact about this diffeomorphism g. It preserves all of the structures on Newtonian spacetime *except* λ^a. It satisfies $g^*(t_a) = t_a$ and $g^*(h^{ab}) = h^{ab}$, and in addition, it preserves ∇, in the sense that $g_*(\nabla) = \nabla$. But the boost symmetry g does not satisfy $g^*(\lambda^a) = \lambda^a$, which we can easily see since g maps trajectories that are at rest to trajectories that are not at rest. This is the essential characteristic of the symmetry g that makes the boost argument show that Newtonian spacetime is a non-ideal spacetime setting: g is *not an isomorphism* of the underlying spacetime structure. It preserves all of the structures on Newtonian spacetime except for the one that the boost argument aims to show is 'unscientific' and 'bad'.

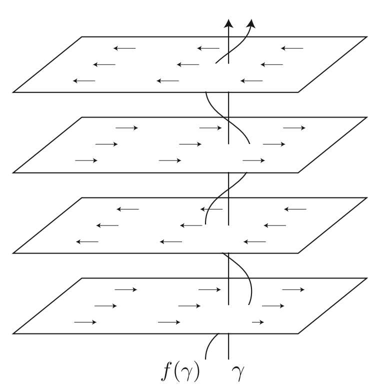
There is a sense in which this is a necessary condition that a symmetry argument must satisfy if it is to demonstrate that a particular spacetime is a non-ideal setting for our theory of motion. The symmetry that the argument exhibits must be a map that is *not* an isomorphism of the underlying spacetime structure. We will here give two explanations for why this is a necessary condition on the cogency of a symmetry argument.

First, we consider how the boost symmetry g shows us that Newtonian spacetime is non-ideal. One begins by judging that g preserves all of the structures that we take to be significant.[13] Philosophers might disagree about why we make this judgment. We might, for example, make this judgment because the difference between theories related by g is 'undetectable' (Dasgupta, 2016). Or we might make this judgment because g is a 'dynamical symmetry', in the sense that it maps dynamically allowable trajectories to dynamically allowable trajectories (Earman, 1989, p. 45–6). The reason why we judge that g preserves all of the structures that we care about, however, is not important for our present purposes. All that is important here is that once we decide that g preserves all of the structures that we care about, we have decided that it *should* be considered an isomorphism of the spacetime that underlies our theory of motion. If Newtonian spacetime provides us with our best description of the structure of spacetime, then, as we remarked above, g is not an isomorphism.

So the fact that g is not an isomorphism shows us that there is a mismatch between the structure of Newtonian spacetime and the structure that we want to ascribe to spacetime. If we are to consider g to be an isomorphism, then we need to move to a 'less structured' spacetime. Galilean spacetime, which does not come equipped with a standard of rest and therefore allows g to be an isomorphism, therefore better represents the structure that we want to ascribe to spacetime. It is in this sense that g *not* being an isomorphism from Newtonian spacetime to itself is crucial. It is precisely this feature of g that allows us to argue that Newtonian spacetime is not the ideal setting for our classical theory of motion. In brief, g is not an isomorphism if Newtonian spacetime is our underlying spacetime, and since we *want* g to be an isomorphism, Newtonian spacetime is not an ideal spacetime setting.

We now turn to the second explanation for why a symmetry must *not* be an isomorphism in order for the symmetry argument to motivate a move to a new spacetime setting. This reason concerns the relationship between symmetry and structure. As we remarked above, there is a strong sense in which objects that admit *more* structure-preserving maps between them have *less* structure. This idea about amounts of structure provides another explanation of our necessary condition. We decide that we want g to be an isomorphism of the spacetime that underlies our theory of motion because it preserves all of the structures that we have deemed significant. Since it is *not* an isomorphism from Newtonian spacetime to itself, this means that if we move to a spacetime setting where g is an isomorphism, then we will have moved to a spacetime that admits *more* structure-preserving maps. And therefore the basic idea about amounts of structure suggested above implies that our new spacetime setting will have less structure than Newtonian spacetime. And this captures a sense in which Newtonian spacetime is not an ideal setting for our classical theory of motion: it has more structure than we want our spacetime to have.[14] Note that if g were an isomorphism, then this argument would not go through.

With our necessary condition now in hand, we turn to the curvature argument. The symmetry at the heart of the curvature argument is of an importantly different character than the boost symmetry g, despite the fact that it initially appears to satisfy our necessary condition. Recall how the curvature argument proceeds. Given that we take Galilean spacetime to best represent the underlying structure of spacetime, there seem to be two theories about this body that are both compatible with the empirical data: according to T_1 the body is traversing a straight trajectory, and according to T_2 the body is traversing a curved trajectory because T_2 posits a different law of motion. We can picture the symmetry that relates the theories T_1 and T_2 as follows.



12 See Barrett (2015b) for a more precise formal presentation.

13 This is, of course, an idea that we might reconsider, especially if we come up with some independent reason to think that λ^a itself is significant. As Friedman (1983, p. 115) remarks, "it may be perfectly useful and respectable *in conjunction with other theories*" like classical electromagnetism.

14 Earman (1989) and Friedman (1983) express roughly this same thought.

As with the boost symmetry, our 'curvature symmetry' can be generated by 'flowing' along the vector field in the figure. Formally, the curvature symmetry is the diffeomorphism $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ that we defined above. One can think of the curvature symmetry f as once again 'beveling the deck', but this time doing so unevenly. At first glance, it looks like f satisfies our necessary condition. The diffeomorphism f preserves t_a and h^{ab} – as demonstrated in the proof of Lemma 2 – in the sense that $f^*(t_a) = t_a$ and $f^*(h^{ab}) = h^{ab}$. But f does not preserve ∇, as we see from the fact that it maps geodesics to curves that are not geodesics. At first glance, therefore, it seems like the curvature symmetry f satisfies our necessary condition: it is a map from Galilean spacetime to itself that is not an isomorphism. And like the boost symmetry, it preserves all of the spacetime structures *except for* the one that the argument aims to show is 'unscientific' and 'bad'.

But this first glance is misleading. As we argued in the preceding sections, it is disingenuous to say that Galilean spacetime $(\mathbb{R}^4, t_a, h^{ab}, \nabla)$ represents the spacetime structure that the theory T_2 posits. Indeed, the underlying spacetime of T_2 is either $(\mathbb{R}^4, t_a, h^{ab}, \nabla, f_*(\nabla))$ or $(\mathbb{R}^4, t_a, h^{ab}, f_*(\nabla))$. The first case is a non-starter, since as we argued in Section 3 that spacetime posits strictly more structure than is necessary.[15] In the second case, the curvature symmetry f actually *is* an isomorphism, unlike the boost symmetry. The map f now preserves all of the structures of the underlying spacetimes of T_1 and T_2. We already remarked that it preserves the temporal metric t_a and the spatial metric h^{ab}. But it also preserves the derivative operator since f trivially pushes forward the derivative operator of Galilean spacetime to the derivative operator $f_*(\nabla)$ that our second spacetime employs. So the curvature symmetry does not satisfy our necessary condition.

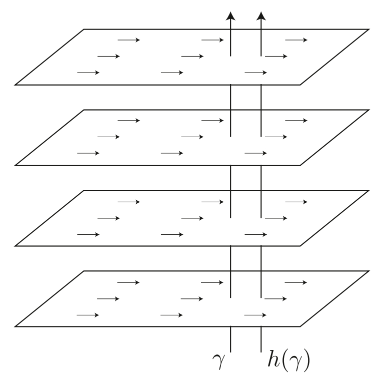
We can see this in the following manner as well. In the case of the boost argument, we were comparing two models of classical physics set in Newtonian spacetime: $(\mathbb{R}^4, t_a, h^{ab}, \nabla, \lambda^a, \gamma)$ and $(\mathbb{R}^4, t_a, h^{ab}, \nabla, \lambda^a, g(\gamma))$, where γ is the trajectory of some particle that is at rest in the first model. We fix the state of rest and consider applying the boost symmetry to the particle. These two models that are non-isomorphic and yet 'observationally' or 'dynamically' equivalent, and therefore we want them to be isomorphic. In the case of the curvature argument – so long as we are understanding T_2 as formulated on $(\mathbb{R}^4, t_a, h^{ab}, f_*(\nabla))$ – we are again comparing two models of classical physics. The curvature argument requires us to consider force free trajectories in both T_1 and T_2. So the models we are comparing must be $(\mathbb{R}^4, t_a, h^{ab}, \nabla, \gamma)$ and $(\mathbb{R}^4, t_a, h^{ab}, f_*(\nabla), f(\gamma))$. And this is the difference between the boost and curvature arguments. These two models are indeed 'observationally' or 'dynamically' equivalent, in the same sense as the models from the boost argument are. But unlike in that case, these models are already isomorphic via the map f.

We therefore cannot use the 'curvature symmetry' f to do the work that we did with the boost symmetry g. Suppose we were to try to argue as in the case of the boost symmetry: We would begin by deciding that f preserves all of the structures that we care about, and so we would *want* it to be an isomorphism of our underlying spacetime. But it already is an isomorphism. Unlike in the case of the boost symmetry g, we do not need to excise any structure from our underlying spacetime in order to make this so.[16]

6. Conclusion

Of course, we have already argued that the curvature argument does not succeed. But this discussion helps to isolate why it is conceptually different from the boost argument. And moreover, recognizing this necessary condition on the cogency of a symmetry argument allows us to reconsider other symmetry arguments too. We will here discuss two other arguments that Dasgupta (2015) mentions: the shift argument in the context of Newtonian spacetime and the boost argument in the context of Galilean spacetime.[17]

The **shift argument** asks us to imagine a situation in which we observe a body that is moving free of forces. Given that we take Newtonian spacetime to best represent the underlying structure of spacetime, there are two theories about this body that are compatible with the empirical data: according to T_1 the body is traveling along the trajectory γ in the below figure, and according to T_2 the body is traveling along the different trajectory $h(\gamma)$. These two theories are related by a 'shift symmetry', which one can picture as follows.



As before, the shift symmetry h is represented by the small horizontal arrows that appear on each of the simultaneity slices. A shift symmetry is defined by 'flowing' each spacetime point along the vector field that the small horizontal arrows represent, so it intuitively shifts each point in $\mathbb{R}^4$ a fixed amount 'to the right'. Now one can easily verify that such a map h is actually an isomorphism from Newtonian spacetime to itself. It preserves the spatial metric, temporal metric, derivative operator, and the standard of rest. While the shift argument may be a problem for various 'substantivalist' theses about the metaphysical status of Newtonian spacetime, it should not prompt us to look for *another* spacetime setting for classical physics. It is unlike the boost argument in this regard.

For our final example we return to the boost argument, but now we consider it in the context of Galilean spacetime. Dasgupta claims that the boost argument also poses a problem for 'Galilean substantivalism', "the view that Galilean space–time exists, and has this structure, independently of its material constituents" (Dasgupta, 2015, p. 613). Our

[15] Moreover, while it is true that f is not an isomorphism from Galilean spacetime to the spacetime $(\mathbb{R}^4, t_a, h^{ab}, \nabla, f_*(\nabla))$, this is because there can be no isomorphism between those two objects. They are completely different kinds of objects – the one has only one derivative operator, while the other has two – and it only makes sense to talk about isomorphisms between objects of the same 'kind'. So unlike in the case of the boost symmetry g, we cannot coherently 'want' this map to be an isomorphism. In this sense the first version of the curvature argument is of a completely different character than the boost argument.

[16] One can imagine using something like Dasgupta's curvature argument to motivate the Saunders (2013) claim that Maxwell spacetime is a better setting for Newton's theory of motion than Galilean spacetime is. For our purposes here, it will suffice to note that in order to do this, one would have to formulate T_2 in a different way than Dasgupta does; his formulation T_2 makes it so that g does not satisfy the necessary condition. In particular, one would have to state Newton's law in such a way that it only appeals to the class of 'non-rotating' derivative operators that Maxwell spacetime employs.

[17] Dasgupta (2015) also considers the shift argument in the context of Galilean spacetime, and our same concern applies there. He is certainly not the only one to consider these arguments, though he does admit that there is a growing consensus among philosophers of physics that shift arguments against Galilean spacetime are not compelling, while boost arguments against Newtonian spacetime are Dasgupta (2015, p. 615). Perhaps the most famous example of a symmetry argument that fails our necessary condition above is the 'hole argument' in the context of general relativity, which is often thought to show that general relativity comes equipped with some 'surplus' structure. See Bradley and Weatherall (2020) and Weatherall (2017) for related discussion. But we will keep our focus here on the case of symmetries in classical physics.

discussion here shows that the boost argument does not pose a problem for Galilean substantivalism in the same way as it posed a problem for the analogous view about Newtonian spacetime. In particular, it does not show that Galilean spacetime comes equipped with some structure that is unscientific and bad. And it should not prompt us to look for a better spacetime setting than Galilean spacetime. This is because, like the curvature argument, the boost argument against Newtonian spacetime fails to satisfy our necessary condition. As we mentioned above, the boost symmetry g *is* an isomorphism from Galilean spacetime to itself. Symmetry arguments in which the exhibited symmetry is already an isomorphism of the underlying spacetime should not prompt us to change our underlying spacetime theory like the boost argument against Newtonian spacetime did. Such arguments do not give us reason to try to excise a piece of structure.

Appendix

The purpose of this appendix is to provide proofs of the lemmas and propositions in the paper. We take them in the order in which they appear.

We begin, however, by remarking that these results – in particular, Propositions 1 and 2 – can likely be strengthened. These two propositions isolate two senses in which the derivative operator $f_*(\nabla)$ is definable from the structure of Galilean spacetime plus the class C of privileged curves. One naturally wonders whether the full structure of Galilean spacetime is needed to do this. It seems that it is not. If we only had the structure of Maxwell spacetime, for example, we might argue as follows.[18] This spacetime has sufficient structure to say when a curve γ' is at rest relative to γ. So we can extend the given curve γ to the class C of curves 'at rest' relative to γ – even though we cannot use ∇ to do so, as we do above – and then consider their tangent fields to define ξ^a. Lemma 3 will therefore go through on Maxwell spacetime. Lemma 4 will follow in the same way as below. Then $f_*(\nabla)$ will be definable from that metric, as in Proposition 1, and since explicit definability generally entails implicit definability, the analogue of Proposition 2 will go through with respect to Maxwell spacetime as well. The point here is that in the argument below, the structure ∇ is only appealed to when we extend γ to the class of curves C and define ξ^a, and it seems that we do not need to appeal to the full structure of ∇ even to do that. This helps to make more precise the minimal role that ∇ plays in T_2.

Whether these propositions hold for an even weaker spacetime structure, like Leibnizian spacetime, remains to be seen. For our purposes here, we discuss in detail only the case of Galilean spacetime, since that is the concern of Dasgupta's curvature argument. We now turn to proofs of our main results.

Lemma 1. *$g_*(\nabla)$ is a derivative operator on N. Moreover, a smooth curve γ' is a geodesic of ∇ if and only if $g\circ\gamma'$ is a geodesic of $g_*(\nabla)$.*

Proof. One easily verifies that $g_*(\nabla)$ is indeed a derivative operator by checking that conditions DO1–DO6 of Malament (2012, p. 49) hold of it. For example, DO5 requires us to verify that for all smooth scalar fields α on N and all smooth vector fields ξ^n on N, $\xi^n g_*(\nabla_n)\alpha = \xi(\alpha)$. We compute the following:

$$\begin{aligned}\xi^n g_*(\nabla_n)\alpha = \xi^n g_*(\nabla_n g^*(\alpha)) &= \xi^n g_*(\nabla_n(\alpha\circ g))\\ &= g^*(\xi^n)\nabla_n(\alpha\circ g)\\ &= g^*(\xi^n)(\alpha\circ g)\\ &= \xi^n(\alpha\circ g\circ g^{-1}) = \xi(\alpha)\end{aligned}$$

The first equality follows from the definition of $g_*(\nabla)$. The second, third, and fifth follow from familiar properties of the pushforward and pullback maps g_* and g^*. The fourth holds since ∇ is a derivative operator and therefore satisfies DO5, and the sixth follows immediately from the fact that $g\circ g^{-1}$ is the identity map.

Now suppose that γ' is a geodesic of ∇ with tangent field ξ^a. Consider the curve $g\circ\gamma'$ with tangent field $g_*(\xi^a)$. Let v_a be a covector at a point $p = g\circ\gamma'(s)$. We compute:

$$\begin{aligned}g_*(\xi^n)g_*(\nabla_n)g_*(\xi^a)\Big|_p \cdot v_a &= g_*(\xi^n)g_*(\nabla_n g^*(g_*(\xi^a)))\Big|_p \cdot v_a\\ &= g_*(\xi^n)g_*(\nabla_n\xi^a)\Big|_p \cdot v_a\\ &= g_*(\xi^n\nabla_n\xi^a)\Big|_p \cdot v_a\\ &= \xi^n\nabla_n\xi^a\Big|_{g^{-1}(p)} \cdot g^*(v_a)\\ &= 0\end{aligned}$$

The first equality follows from the definition of $g_*(\nabla)$, the second from the fact that $g^*\circ g_*$ is the identity map, the third and fourth from properties of the pushforward, and the fifth since γ' is a geodesic of ∇. Since v_a was arbitrary, this means that $g\circ\gamma'$ is a geodesic of $g_*(\nabla)$. An analogous computation shows that if $g\circ\gamma'$ is a geodesic of $g_*(\nabla)$ then γ' is a geodesic of ∇. □

Lemma 2. *$f_*(\nabla)$ is compatible with both t_a and h^{ab}, in the sense that $f_*(\nabla_n)t_a = \mathbf{0}$ and $f_*(\nabla_n)h^{ab} = \mathbf{0}$.*

Proof. One computes all of the following:

$$\begin{aligned}&f^*(d_n x^j) = d_n x^j \text{ for } j \neq 2\\ &f^*(d_n x^2) = \cos(x^1)d_n x^1 + d_n x^2\\ &f^*(\frac{\partial}{\partial x^i})^a = (\frac{\partial}{\partial x^i})^a \text{ for } i \neq 1\\ &f^*(\frac{\partial}{\partial x^1})^a = (\frac{\partial}{\partial x^1})^a - \cos(x^1)(\frac{\partial}{\partial x^2})^a\end{aligned} \tag{4}$$

It immediately follows from (4) that $f^*(t_a) = t_a$ and $f^*(h^{ab}) = h^{ab}$. This in turn implies that

$$f_*(\nabla_n)(t_a) = f_*(\nabla_n f^*(t_a)) = f_*(\nabla_n t_a) = \mathbf{0}$$

The last equality follows from the fact that ∇ is compatible with t_a. One shows that and $f_*(\nabla_n)(h^{ab}) = \mathbf{0}$ in a perfectly analogous manner. □

Lemma 3. *If $\gamma' : \mathbb{R} \to \mathbb{R}^4$ is a smooth curve with tangent field ξ^a that satisfies $t_a\xi^a = 1$, then there is a unique vector field that agrees with ξ^a on the image of γ' and is constant along all spacelike curves.*

Proof. We first demonstrate existence. The structure of Galilean spacetime provides us with a natural way to 'extend' ξ^n to a field on all of $\mathbb{R}^4$. Given a point $p \in \mathbb{R}^4$ we consider a spacelike curve that passes through the point p and $\gamma'(s_0)$ for some $s_0 \in \mathbb{R}$. That such a spacelike curve exists is guaranteed by the fact that $t_a\xi^a = 1$, so γ' intersects every simultaneity slice, and in particular, the one that p lies on. We parallel transport the vector $\xi^n|_{\gamma'(s_0)}$ along this spacelike curve to the point p to obtain the vector $\xi^n|_p$. Since ∇ is flat, parallel transport is path independent, and our choice of spacelike curve makes no difference. And moreover the spacelike curve will intersect γ' exactly once, so $\xi^n|_p$ is well-defined. Because ξ^a results from this procedure of parallel transporting along spacelike curves, it is immediate that ξ^a is constant along all spacelike curves.

Suppose now that α^n and β^n agree with ξ^n on the image of γ' and are constant along all spacelike curves. Let $p \in \mathbb{R}^4$ and consider the point $\gamma'(s)$ that lies on the same simultaneity slice as p, so that $x^1(p) = x^1(\gamma'(s))$. We write out both α^n and β^n in standard coordinates as $\alpha^n = \sum_{i=1}^4 \overset{i}{\alpha}(\frac{\partial}{\partial x^i})^n$ and $\beta^n = \sum_{i=1}^4 \overset{i}{\beta}(\frac{\partial}{\partial x^i})^n$. We know that $\overset{i}{\alpha}(\gamma'(s)) = \overset{i}{\beta}(\gamma'(s))$ for each i since the two fields agree on the image of γ'. Now for each $j \neq 1$ we compute the following:

$$\mathbf{0} = (\frac{\partial}{\partial x^j})^n\nabla_n\alpha^a$$

[18] See Earman (1989) for a definition of Maxwell spacetime.

$$= (\frac{\partial}{\partial x^j})^n \nabla_n (\sum_{i=1}^{4} \overset{i}{\alpha} (\frac{\partial}{\partial x^i})^a)$$
$$= (\frac{\partial}{\partial x^j})^n \sum_{i=1}^{4}\sum_{k=1}^{4} \frac{\partial \overset{i}{\alpha}}{\partial x^k} d_n x^k (\frac{\partial}{\partial x^i})^a$$
$$= \sum_{i=1}^{4} \frac{\partial \overset{i}{\alpha}}{\partial x^j} (\frac{\partial}{\partial x^i})^a$$

The first equality follows since the 'jth-coordinate curve' (that has tangent field $(\frac{\partial}{\partial x^j})^n$) is spacelike and α^n is constant along spacelike curves. The second equality results from writing out α^n in coordinates, the third from the definition of the coordinate derivative operator, and the fourth from the fact that $(\frac{\partial}{\partial x^j})^n d_n x^k = \delta^k_j$. This implies that

$$\frac{\partial \overset{i}{\alpha}}{\partial x^j} = 0 \tag{5}$$

for every i and each $j \neq 1$. We compute in the same manner that $\frac{\partial \overset{i}{\beta}}{\partial x^j} = 0$ for every i and each $j \neq 1$. This means that when written out in standard coordinates the smooth scalar fields $\overset{i}{\alpha}$ and $\overset{i}{\beta}$ are functions of just the x^1-coordinate. Since the x^1-coordinate of p is the same as the x^1-coordinate of $\gamma'(s)$ we immediately see that $\overset{i}{\alpha}(p) = \overset{i}{\alpha}(\gamma'(s))$ and $\overset{i}{\beta}(p) = \overset{i}{\beta}(\gamma'(s))$ for each i. That implies that $\overset{i}{\alpha}(p) = \overset{i}{\beta}(p)$ for all i, which means that $\alpha^n = \beta^n$. □

Lemma 4. *If γ' is a smooth curve in C with tangent field ξ^n, then the smooth tensor field $h^{ab} + \xi^a \xi^b$ is a metric on $\mathbb{R}^4$.*

Proof. It is clear that $h^{ab} + \xi^a \xi^b$ is symmetric since h^{ab} is. Consider the smooth tensor field $\hat{h}_{ab} + t_a t_b$, where $\hat{h}_{ab}$ is the spatial projection field associated with h^{ab} and ξ^a Malament (2012, Prop. 4.1.2). We compute:

$$\begin{aligned}(\hat{h}_{ab} + t_a t_b)(h^{bc} + \xi^b \xi^c) &= \hat{h}_{ab} h^{bc} + \hat{h}_{ab}\xi^b\xi^c + t_a t_b h^{bc} + t_a t_b \xi^b \xi^c \\ &= \hat{h}_{ab} h^{bc} + \hat{h}_{ab}\xi^b\xi^c + t_a \xi^c \\ &= \delta^c_a - t_a\xi^c + t_a \xi^c \\ &= \delta^c_a\end{aligned}$$

The first equality follows from properties of tensor multiplication, the second since $t_b h^{bc} = \mathbf{0}$ and $t_b \xi^b = 1$, and the third from the definition of $\hat{h}_{ab}$. So $h^{ab} + \xi^a \xi^b$ is symmetric and invertible (i.e. non-degenerate) and therefore a metric. □

Proposition 1. *Let $\gamma' : \mathbb{R} \to \mathbb{R}^4$ be a smooth curve in C with tangent field ξ^n. Then $f_*(\nabla)$ is the unique derivative operator compatible with the metric $h^{ab} + \xi^a \xi^b$.*

Proof. We need to show that $f_*(\nabla_n)(h^{ab}+\xi^a\xi^b) = \mathbf{0}$. We begin by writing out $\xi^a = \sum_{i=1}^4 \overset{i}{\xi}(\frac{\partial}{\partial x^i})^a$ in standard coordinates. One then computes in the same manner as in the proof of Eq. (5) in Lemma 3 that

$$\frac{\partial \overset{i}{\xi}}{\partial x^j} = 0 \tag{6}$$

for every i and each $j \neq 1$. Note also that because $t_a \xi^a = 1$ and $t_a = d_a x^1$, it must be that $\overset{1}{\xi} = 1$.

Lemma 2 implies that $f_*(\nabla_n)(h^{ab}) = \mathbf{0}$. So if we can show that $f_*(\nabla_n)(\xi^a) = \mathbf{0}$, then we will be done. By the definition of $f_*(\nabla)$, we know that $f_*(\nabla_n)(\xi^a) = f_*(\nabla_n f^*(\xi^a))$. So now we begin to compute.

$$\begin{aligned}\nabla_n f^*(\xi^a) &= \nabla_n (\sum_{i=1}^{4} \overset{i}{\xi} f^*(\frac{\partial}{\partial x^i})^a) \\ &= \nabla_n \left(\overset{1}{\xi}((\frac{\partial}{\partial x^1})^a - \cos(x^1)(\frac{\partial}{\partial x^2})^a) + \sum_{i=2}^{4} \overset{i}{\xi}(\frac{\partial}{\partial x^i})^a \right) \\ &= \nabla_n \left(\sum_{i=1}^{4} \overset{i}{\xi}(\frac{\partial}{\partial x^i})^a - \cos(x^1)(\frac{\partial}{\partial x^2})^a \right) \\ &= \sum_{i=1}^{4} (\nabla_n \overset{i}{\xi})(\frac{\partial}{\partial x^i})^a - (\nabla_n \cos(x^1))(\frac{\partial}{\partial x^2})^a \\ &= \sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial \overset{i}{\xi}}{\partial x^j} d_n x^j (\frac{\partial}{\partial x^i})^a + \sin(x^1) d_n x^1 (\frac{\partial}{\partial x^2})^a \\ &= \sum_{i=1}^{4} \frac{\partial \overset{i}{\xi}}{\partial x^1} d_n x^1 (\frac{\partial}{\partial x^i})^a + \sin(x^1) d_n x^1 (\frac{\partial}{\partial x^2})^a\end{aligned}$$

The first equality follows from writing out ξ^a in standard coordinates and the linearity of f^*, the second from the computation of f^* cataloged in (4), and the third by rearranging terms. The fourth and fifth equalities follow from the definition of the coordinate derivative operator, and the last equality follows from Eq. (6).

This computation then implies the following.

$$\begin{aligned}f_*(\nabla_n f^*(\xi^a)) &= f_*\left(\sum_{i=1}^{4} \frac{\partial \overset{i}{\xi}}{\partial x^1} d_n x^1 (\frac{\partial}{\partial x^i})^a + \sin(x^1) d_n x^1 (\frac{\partial}{\partial x^2})^a \right) \\ &= \sum_{i=1}^{4} \frac{\partial \overset{i}{\xi}}{\partial x^1} f_*(d_n x^1) f_*(\frac{\partial}{\partial x^i})^a + \sin(x^1) f_*(d_n x^1) f_*(\frac{\partial}{\partial x^2})^a \\ &= \sum_{i=1}^{4} \frac{\partial \overset{i}{\xi}}{\partial x^1} d_n x^1 (\frac{\partial}{\partial x^i})^a + \frac{\partial \overset{1}{\xi}}{\partial x^1} \cos(x^1)(\frac{\partial}{\partial x^2})^a \\ &\quad + \sin(x^1) d_n x^1 (\frac{\partial}{\partial x^2})^a \\ &= \sum_{i=1}^{4} \frac{\partial \overset{i}{\xi}}{\partial x^1} d_n x^1 (\frac{\partial}{\partial x^i})^a + \sin(x^1) d_n x^1 (\frac{\partial}{\partial x^2})^a\end{aligned}$$

The first equality follows from our first computation, the second from properties of the pushforward f_*, and the third from the computation of f^* cataloged in (4) and the fact that f_* is inverse to it. The fourth follows since $\frac{\partial \overset{1}{\xi}}{\partial x^1} = 0$, which we know because $\overset{1}{\xi} = 1$.

Putting all of this together yields the following equation:

$$f_*(\nabla_n)(\xi^a) = \sum_{i=1}^{4} \frac{\partial \overset{i}{\xi}}{\partial x^1} d_n x^1 (\frac{\partial}{\partial x^i})^a + \sin(x^1) d_n x^1 (\frac{\partial}{\partial x^2})^a \tag{7}$$

Now we want to show that the right-hand side of Eq. (7) is $\mathbf{0}$. Recall that $\xi^n \nabla_n \xi^a = -\sin(x^1)(\frac{\partial}{\partial x^2})^a$ since γ' is in C. Writing out the left-hand side of this equation in standard coordinates gives us that

$$\sum_{i=1}^{4}\sum_{j=1}^{4} \overset{j}{\xi} \frac{\partial \overset{i}{\xi}}{\partial x^j} \left(\frac{\partial}{\partial x^i}\right)^a = -\sin(x^1)\left(\frac{\partial}{\partial x^2}\right)^a \tag{8}$$

Unraveling equation (8) then gives us all of the following.

$$\frac{\partial \overset{1}{\xi}}{\partial x^1} = 0 \qquad \frac{\partial \overset{2}{\xi}}{\partial x^1} = -\sin(x^1) \qquad \frac{\partial \overset{3}{\xi}}{\partial x^1} = 0 \qquad \frac{\partial \overset{4}{\xi}}{\partial x^1} = 0$$

For example, in the $i = 2$ case, Eq. (8) implies that $\sum_{j=1}^4 \overset{j}{\xi} \frac{\partial \overset{2}{\xi}}{\partial x^j} = -\sin(x^1)$. Eq. (6) then implies that all of the terms in the summation are 0 except for the $j = 1$ term, so $\overset{1}{\xi} \frac{\partial \overset{2}{\xi}}{\partial x^1} = -\sin(x^1)$. Since $\overset{1}{\xi} = 1$, this means that $\frac{\partial \xi^2}{\partial x^1} = -\sin(x^1)$, as desired. Now plugging all of these back into Eq. (7) gives us that $f_*(\nabla_n)(\xi^a) = \mathbf{0}$. And that immediately implies that $f_*(\nabla_n)(h^{ab} + \xi^a\xi^b) = \mathbf{0}$, so $f_*(\nabla)$ is the unique derivative operator compatible with the metric $h^{ab} + \xi^a \xi^b$. □

Proposition 2. *Let $g : \mathbb{R}^4 \to \mathbb{R}^4$ be an automorphism of Galilean spacetime such that a smooth curve γ' is in C if and only if $g \circ \gamma'$ is in C. Then $g_*(f_*(\nabla)) = f_*(\nabla)$.*

Proof. Let γ' be a curve in C with tangent field ξ^a. We will show that $g_*(f_*(\nabla))$ is compatible with the metric $h^{ab} + \xi^a\xi^b$, and then Proposition 1 will imply that $g_*(f_*(\nabla)) = f_*(\nabla)$. We begin by computing.

$$g_*(f_*(\nabla))(h^{ab} + \xi^a\xi^b) = g_*\left(f_*(\nabla)(g^*(h^{ab}) + g^*(\xi^a)g^*(\xi^b))\right)$$

$$= g_*\big(f_*(\nabla)(g^*(\xi^a)g^*(\xi^b))\big)$$

The first equality follows from the definition of $g_*(f_*(\nabla))$, the second from Lemma 2 and the fact that $g^*(h^{ab}) = h^{ab}$. This means that if we can show that $f_*(\nabla)(g^*(\xi^a)) = \mathbf{0}$, then we have the desired result. We turn to that now.

We have assumed that a curve is in C if and only if its image under g is. It follows from this that $g^{-1}\circ\gamma'$ is in C. This curve has tangent field $g^*(\xi^a)$ along its image $g^{-1}\circ\gamma'[\mathbb{R}]$. Now there is a slight subtlety that arises here: We need to verify that $g^*(\xi^a)$ – thought of as the tangent field to $g^{-1}\circ\gamma'$ extended to all of $\mathbb{R}^4$ using Lemma 3 – is the same as $g^*(\xi^a)$ — the pullback of the extension of ξ^a from $\gamma'[\mathbb{R}]$ to all of $\mathbb{R}^4$. In order to do so, we have to show that the latter agrees with the former on the image of $g^{-1}\circ\gamma'$ and that it is constant along spacelike curves. The uniqueness clause from Lemma 3 will then imply that the two are the same field.

First, let $\alpha : \mathbb{R}^4 \to \mathbb{R}$ be a smooth scalar field and $s \in \mathbb{R}$. Then we compute the following.

$$\begin{aligned} g^*(\xi^a)\cdot\alpha\Big|_{g^{-1}\circ\gamma'(s)} &= \xi^a\cdot(\alpha\circ g^{-1})\Big|_{\gamma'(s)} \\ &= \overrightarrow{\gamma'}_{\gamma'(s)}\cdot(\alpha\circ g^{-1}) \\ &= \frac{d}{ds}(\alpha\circ g^{-1}\circ\gamma')(s) \\ &= \overrightarrow{(g^{-1}\circ\gamma')}_{g^{-1}\circ\gamma'(s)}\cdot\alpha \end{aligned}$$

The first equality follows from the definition of the pullback g^*, the second from the fact that ξ^a is the tangent field to γ', and the third and fourth from the definition of the tangent vector to a curve Malament (2012, p. 11). This means that the pullback $g^*(\xi^a)$ has the same action on an arbitrary smooth scalar field as the tangent field to $g^{-1}\circ\gamma'$ does, and so the two are equal on the image of $g^{-1}\circ\gamma'$.

Second, we show that the pullback $g^*(\xi^a)$ is constant along spacelike curves. Let β be a spacelike curve with tangent field λ^a. We need to show that $\lambda^n\nabla_n g^*(\xi^a) = \mathbf{0}$. So we compute.

$$\lambda^n g^*(\nabla_n\xi^a) = \lambda^n g^*(g_*(\nabla_n)(\xi^a)) = \lambda^n g^*(g_*(\nabla_n g^*(\xi^a))) = \lambda^n\nabla_n g^*(\xi^a) \tag{9}$$

The first equality holds since $g_*(\nabla) = \nabla$. The second follows by unraveling the definition of $g_*(\nabla)$, and the third from the fact that $g^*\circ g_*$ is the identity. We now note the following:

$$g_*(\lambda^n g^*(\nabla_n\xi^a)) = g_*(\lambda^n)\nabla_n\xi^a = \mathbf{0}$$

The first equality follows since $g_*\circ g^*$ is the identity. Since g preserves t_a, it must be that $g_*(\lambda^n)$ is spacelike, so since ξ^a is constant along spacelike curves (by Lemma 3), the second equality follows. Since g is a diffeomorphism, g_* is bijective, so it must be that $\lambda^n g^*(\nabla_n\xi^a) = \mathbf{0}$. So Eq. (9) then implies that $g^*(\xi^a)$ is constant along β.

This means that we can treat $g^*(\xi^a)$ exactly like ξ^a in the proof of Proposition 1. $g^*(\xi^a)$ is the extension (via Lemma 3) to all of $\mathbb{R}^4$ of the tangent field of a curve in C, and in the proof of Proposition 1 ξ^a was one arbitrary such field. So as we did in Proposition 1 when we computed that $f_*(\nabla_n)\xi^a = \mathbf{0}$, we here compute that $f_*(\nabla_n)(g^*(\xi^a)) = \mathbf{0}$, as desired. □

Proposition 3. *Let $\gamma' : \mathbb{R} \to \mathbb{R}^4$ be a smooth curve with tangent field ξ^a that satisfies $t_a\xi^a = 1$. Then $\xi^n f_*(\nabla_n)\xi^a = \xi^n\nabla_n\xi^a + \sin(x^1)\big(\frac{\partial}{\partial x^2}\big)^a$.*

Proof. The proof is a simple computation, which has in effect already been done during the proof of Proposition 1. One begins by computing equation (7). Note that for that computation we only relied on the fact that $t_a\xi^a = 1$, not on the assumption that γ' was in C, so it will go through given our present assumptions as well. We restate the equation here for convenience:

$$f_*(\nabla_n)(\xi^a) = \sum_{i=1}^{4}\frac{\partial\overset{i}{\xi}}{\partial x^1}d_nx^1\Big(\frac{\partial}{\partial x^i}\Big)^a + \sin(x^1)d_nx^1\Big(\frac{\partial}{\partial x^2}\Big)^a \tag{7}$$

We then compute the following, putting the pieces back together:

$$\begin{aligned} \xi^n f_*(\nabla_n)(\xi^a) &= \xi^n\sum_{i=1}^{4}\frac{\partial\overset{i}{\xi}}{\partial x^1}d_nx^1\Big(\frac{\partial}{\partial x^i}\Big)^a + \xi^n\sin(x^1)d_nx^1\Big(\frac{\partial}{\partial x^2}\Big)^a \\ &= \xi^n\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\partial\overset{i}{\xi}}{\partial x^j}d_nx^j\Big(\frac{\partial}{\partial x^i}\Big)^a + \sin(x^1)\Big(\frac{\partial}{\partial x^2}\Big)^a \\ &= \xi^n\sum_{i=1}^{4}(\nabla_n\overset{i}{\xi})\Big(\frac{\partial}{\partial x^i}\Big)^a + \sin(x^1)\Big(\frac{\partial}{\partial x^2}\Big)^a \\ &= \xi^n\nabla_n\left(\sum_{i=1}^{4}\overset{i}{\xi}\Big(\frac{\partial}{\partial x^i}\Big)^a\right) + \sin(x^1)\Big(\frac{\partial}{\partial x^2}\Big)^a \\ &= \xi^n\nabla_n\xi^a + \sin(x^1)\Big(\frac{\partial}{\partial x^2}\Big)^a \end{aligned}$$

The first equality follows from Eq. (7), while the second follows from Eq. (6) and the fact that $d_nx^1\xi^n = 1$. The third and fourth equalities follow from the definition of the coordinate derivative operator. The last equality holds trivially by the way we have written ξ^a in standard coordinates. □

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